Episode 19

Rigid Body Dynamics

ENGN0040: Dynamics and Vibrations
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Topics for todays class

Analyzing motion of rigid bodies

1. Rotational forces – ‘Pure moments (couples, torques)’
2. Equations of motion for rigid bodies (translation & rotation)
3. Examples
4. Short-cut for analyzing rigid bodies rotating about a fixed point
5. Examples

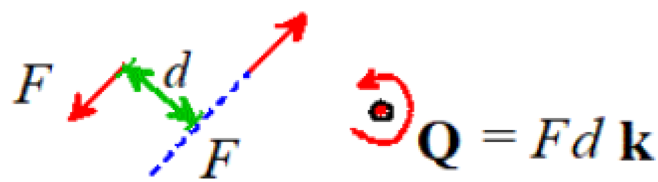
6.5 Analyzing Rigid Body Motion

6.5.1 "Pure Moments" (Couples / Torques)

Definition: Pure moment $\underline{Q}$ = generalized force that rotates a rigid body without moving COM

$\underline{Q}$ is a vector : magnitude Nm
Direction parallel to $d\underline{h}/dt$

Two equal & opposite forces are equivalent to a pure moment



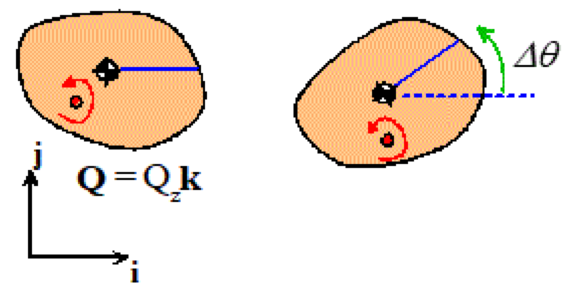
In 2D $\underline{Q}$ is always in $\underline{k}$ direction

Power of a pure moment

$P = \underline{Q} \cdot \underline{\omega}$

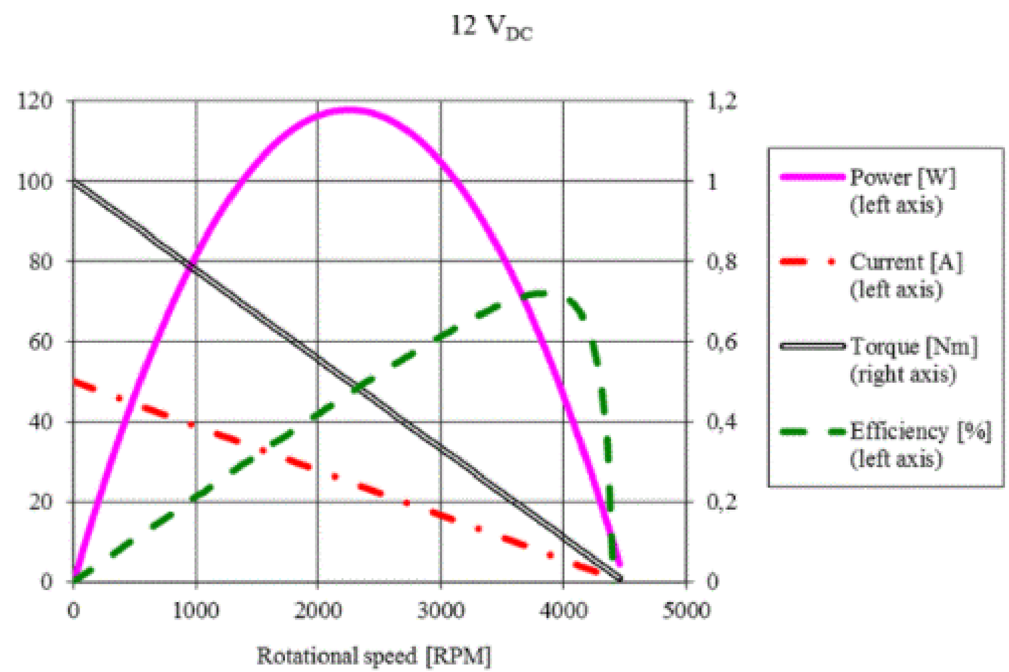
Work $\quad W = \int_0^t P\,dt$

In 2D $\quad W = \int_0^{\Delta\theta} Q(\theta)\,d\theta$



Motor shaft applies a pure moment (torque)
Quantified by "Torque Curve"

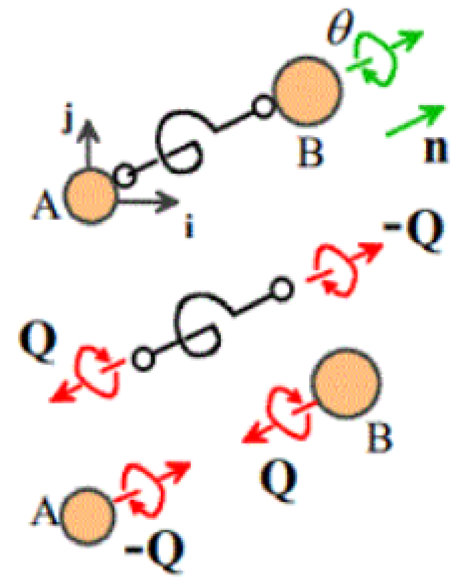
Example: Torque and power curves for small electric motor

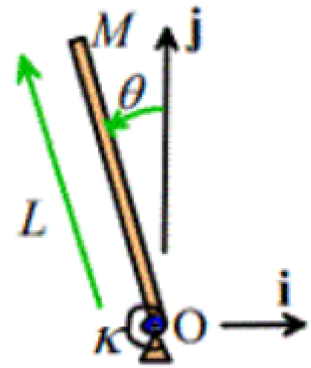


Torsional Springs

Exert a restoring moment proportional to twist

$\underline{Q} = -\kappa\theta\,\underline{n}$ κ – "torsional stiffness"

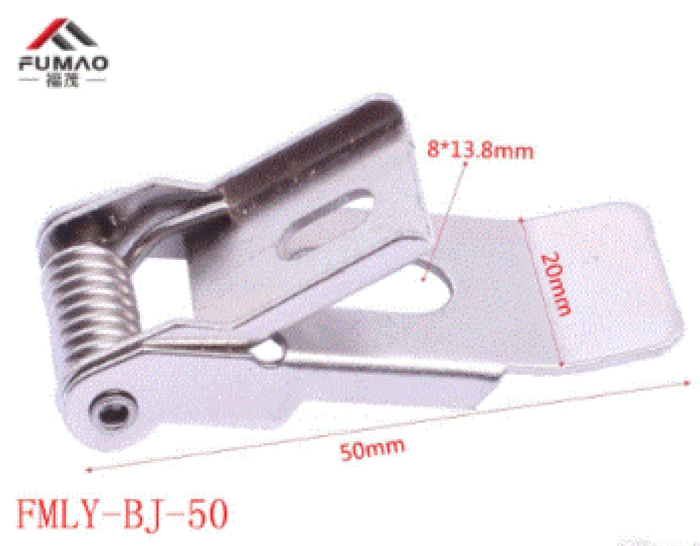




In 2D $\underline{Q} = -\kappa\theta\,\underline{k}$

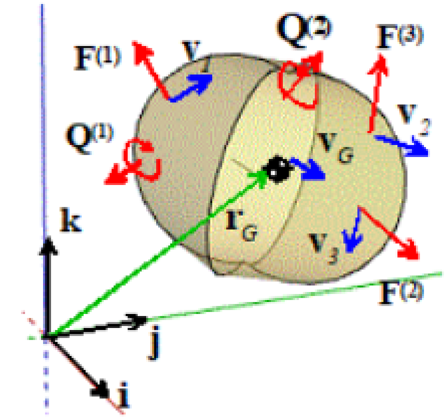
Potential energy $U = \frac{1}{2}\kappa\theta^2$

Example of a torsional spring



6.5.2 Equations of motion for rigid bodies

Linear Momentum



$$\Sigma \underline{F} = d\underline{p}/dt \qquad \underline{p} = M\,\underline{v}_G$$

$$\Rightarrow \Sigma \underline{F} = M\,\underline{a}_G$$

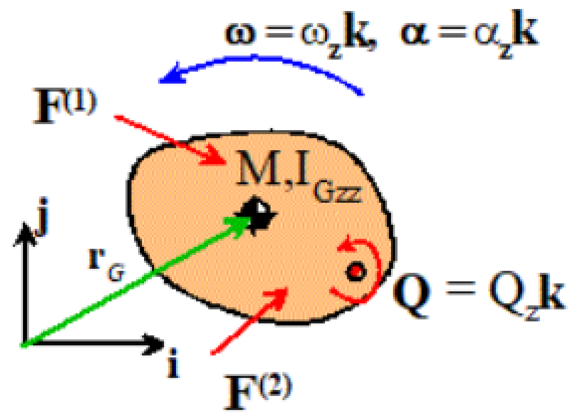
Angular momentum (about O)

$$\Sigma \underline{r} \times \underline{F} + \Sigma \underline{Q} = d\underline{h}_o/dt$$

$$\underline{h}_o = \underline{r}_G \times M\,\underline{v}_G + I_G\,\underline{\omega}$$

$$\Rightarrow \Sigma \underline{r} \times \underline{F} + \Sigma \underline{Q} = \underline{r}_G \times M\,\underline{a}_G + I_G\,\underline{\alpha} + \underline{\omega} \times (I_G\,\underline{\omega})$$

Simplified 2D equations



$$\boxed{\sum \underline{F} = M \underline{a}_G}$$

$$\sum \underline{r} \times \underline{F} + \sum Q_z \underline{k} = d\,\underline{h}_0 / dt$$

$$\underline{h}_0 = \underline{r}_G \times M \underline{v}_G + I_{Gzz} \omega_z \underline{k}$$

$$\Rightarrow \boxed{\sum \underline{r} \times \underline{F} + \sum Q_z \underline{k} = \underline{r}_G \times M \underline{a}_G + I_{Gzz} \alpha_z \underline{k}}$$

Example 6.5.3 : A flywheel energy storage device has the following properties:

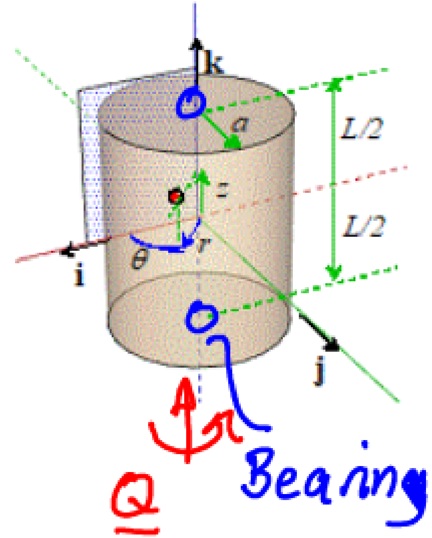
Mass: 2400kg, Radius 1m, Height 2m

It is spun up from rest to 15000rpm (500π rad/s) in 1 min by a constant torque $\mathbf{Q} = Q_z\mathbf{k}$

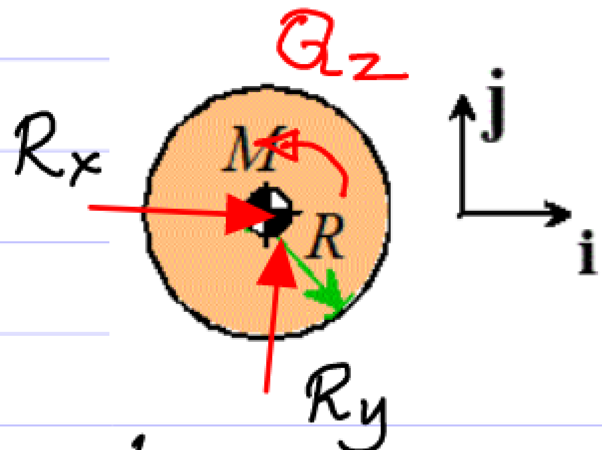
Find: (1) The angular acceleration

(2) The torque

(3) The work done by the torque



Dynamics Equations



$$\sum \underline{F} = M\underline{a}_G \Rightarrow R_x\underline{i} + R_y\underline{j} = \underline{0}$$

$$\Rightarrow R_x = R_y = 0$$

$$\sum \underline{r}\times\underline{F} + \sum Q_z\underline{k} = \overset{=0}{\cancel{\underline{r}_G}} \times M\,\overset{=0}{\cancel{\underline{a}_G}} + I_{Gzz}\alpha_z\underline{k}$$

$$\Rightarrow Q_z = I_{Gzz}\alpha_z$$

Hence α_z is constant

Constant accel formula $\omega_2 = \alpha_2 t$

$$\Rightarrow \boxed{\alpha_2 = \omega_2 / t = 500\pi/60 = 50\pi/6 \ \text{rad/s}^2}$$

Torque: $Q_2 = I_{Gzz}\,\alpha_2$ $\qquad I_{Gzz} = \frac{1}{2} M R^2$

$$= 1200 \ \text{kg m}^2$$

Hence $\boxed{Q_2 = 10^4 \pi \ \text{Nm}}$

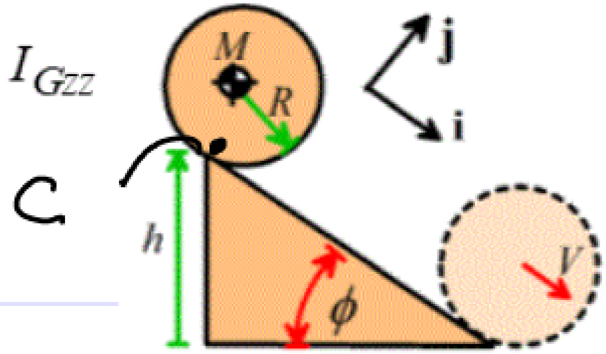
Work: $W = \int_0^t P\,dt$ $\qquad P = Q_2 \omega_2$

$$\Rightarrow W = \int_0^t Q_2 \alpha_2 t = \frac{1}{2} Q_2 \alpha_2 t^2$$

$$\Rightarrow \boxed{W = 1.5 \ \text{GJ}}$$

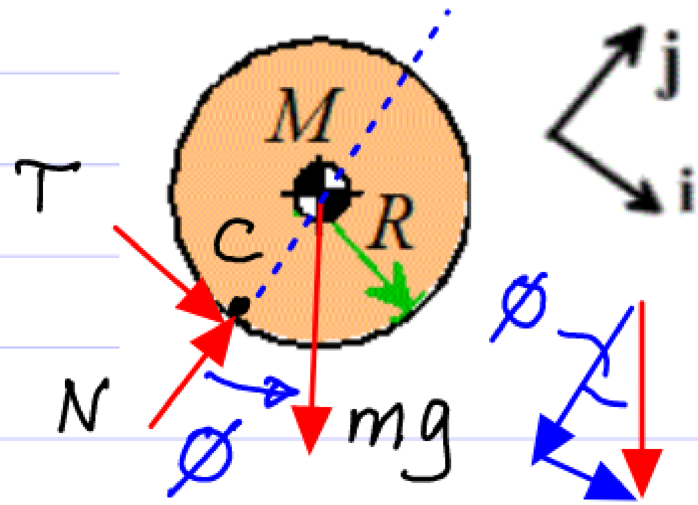
"Giga" $= 10^9$

Example 6.5.4: A solid of revolution, mass M and moment of inertia I_{Gzz} starts from rest at the top of the ramp. Assume no slip. How long does it take to reach the bottom?



Approach: Find accel (dynamics) and integrate

FBD



Apply angular momentum eq about point C

$$\Sigma \underline{r} \times \underline{F} = \underline{r}_G \times M \underline{a}_G + I_{Gzz} \alpha_z \underline{k}$$

$$\Rightarrow R\underline{j} \times (mg\sin\phi\, \underline{i} - mg\cos\phi\, \underline{j}) = R\underline{j} \times M a_{Gx} \underline{i} + I_{Gzz} \alpha_z \underline{k}$$

$$\Rightarrow -Rmg\sin\phi\, \underline{k} = -Rma_{Gx} \underline{k} + I_{Gzz} \alpha_z \underline{k} \qquad (1)$$

Rolling wheel formula (see sect 6.3.6) $\quad a_{Gx} = -\alpha_z R \qquad (2)$

Combine (1) & (2)

$$-Rmg\sin\phi = -Rma_{Gx}\left(1 + I_{Gzz}/R^2\right)$$

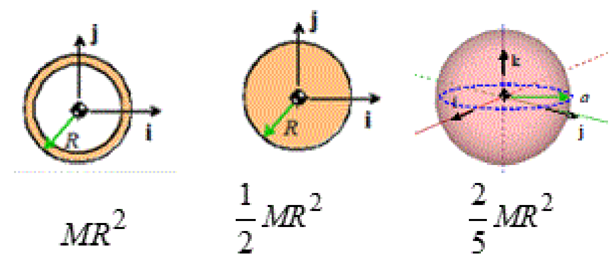
$$\Rightarrow \boxed{a_{Gx} = \frac{g\sin\phi}{1 + I_{Gzz}/R^2}}$$

Const accel formula $\dfrac{h}{\sin\phi} = \frac{1}{2} a_{Gx} t^2$

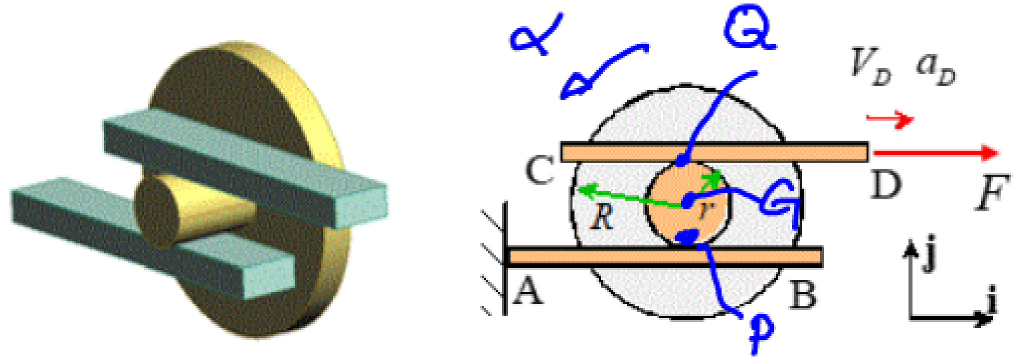
$$\Rightarrow \boxed{t = \sqrt{2h/a_{Gx}\sin\phi} = \frac{1}{\sin\phi}\sqrt{\frac{2h}{g}}\sqrt{1+\frac{I_{Gzz}}{R^2}}}$$

Objects with small I_{Gzz} reach bottom first

$t_{ring} > t_{cylinder} > t_{sphere}$



6.5.6 Example The figure shows a design for an 'inerter'. The flywheel/pinion has mass M and mass moment of inertia I_{Gzz}. Bars AB and CD have mass m. Point A is stationary and point D moves horizontally with acceleration a_D. Find a formula for the force F acting at point D.

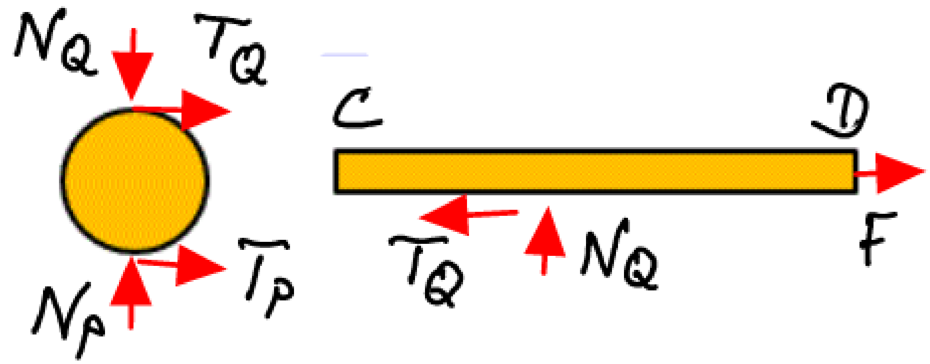


The wheel/bars are rack-and-pinion gears (roll without slip)

Recall (ex 6.3.7) $\alpha = -a_D/2r$

Rolling wheel formula (6.3.6) $a_{Gx} = -\alpha r$

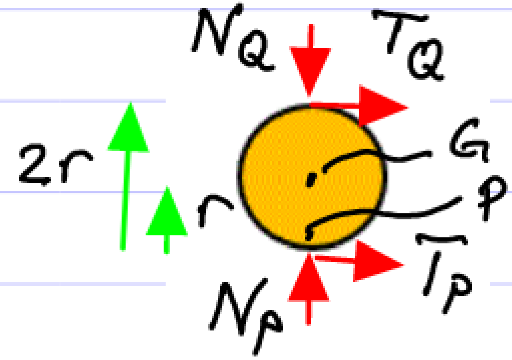
Dynamics



$\underline{F} = m\underline{a}_G$ for CD

$$(F - T_Q)\,\underline{i} + N_Q\,\underline{j} = m a_D\,\underline{i}$$

Angular momentum about P



$$-2rT_Q = -rMa_{Gx} + I_{Gzz}\alpha$$

Eliminate α, a_{Gx}, T_Q

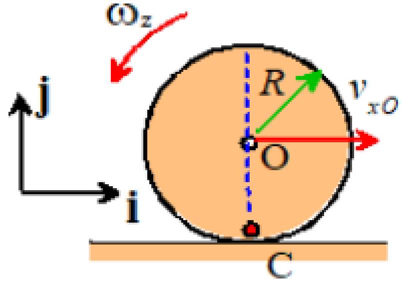
$$T_Q = F - ma_D$$

$$-2r(F - ma_D) = -rM\frac{a_D}{2} - I_{Gzz}\frac{a_D}{2r}$$

Hence $$\boxed{F = \left(m + \frac{M}{4} + \frac{I_{Gzz}}{(2r)^2}\right)a_D}$$

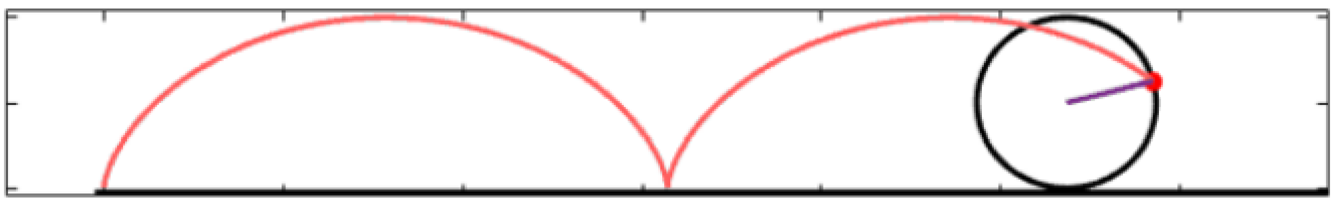
In actual design $\frac{I_{Gzz}}{(2r)^2} \gg m + \frac{M}{4}$

6.5.6 Forces on rolling/sliding wheels

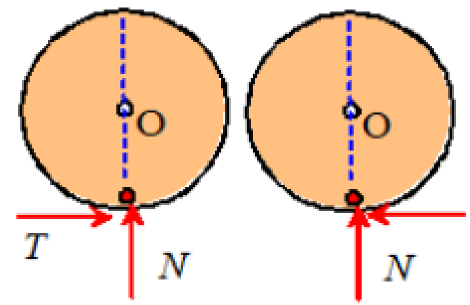


$$\mathbf{v}_C - \mathbf{v}_O = \omega_z \mathbf{k} \times (\mathbf{r}_C - \mathbf{r}_O) \Rightarrow v_{xC} = v_{xO} + \omega_z R$$

Wheel rolling without slip $\mathbf{v}_C = 0$ $\quad v_{xO} + \omega_z R = 0$

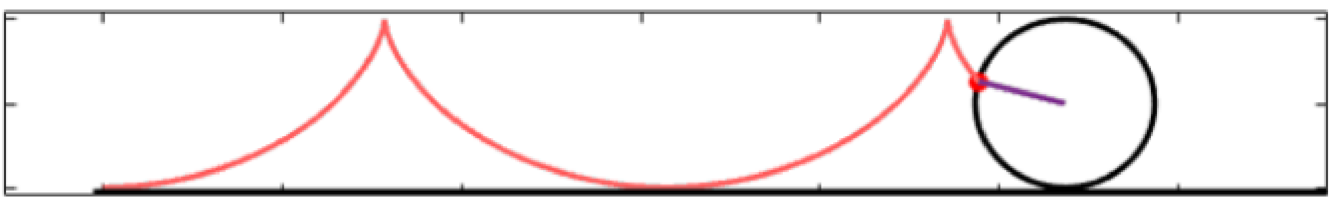


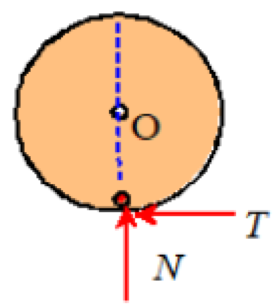
Both FBDs correct



$|T| < \mu N$

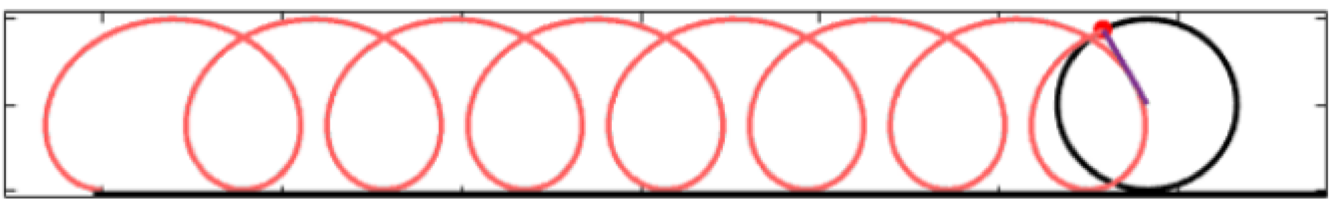
Backspin $v_{xC} > 0$ $\quad v_{xO} + \omega_z R > 0$

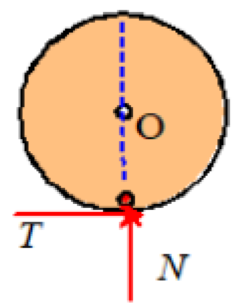




$T = \mu N$

Topspin $v_{xC} < 0$ $\quad v_{xO} + \omega_z R < 0$





$T = \mu N$

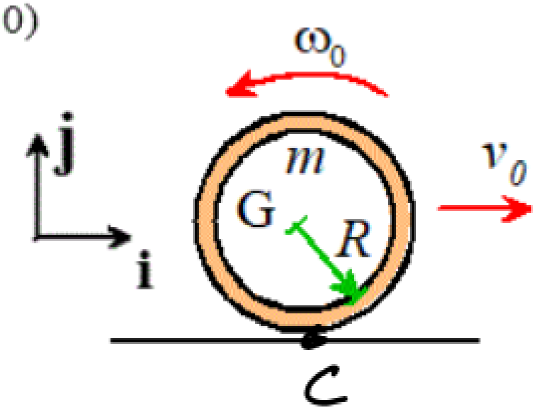
Example 6.5.7 At time *t=0* the center of the ring has velocity $\mathbf{v}_O = V\mathbf{i} \quad (V > 0)$ and angular velocity $\boldsymbol{\omega} = \omega_0 \mathbf{k} \quad (\omega_0 > 0)$ (backspin). Find

(1) The angular velocity as a function of time

(2) The velocity of G as a function of time

(3) The time required for slip to cease at the contact

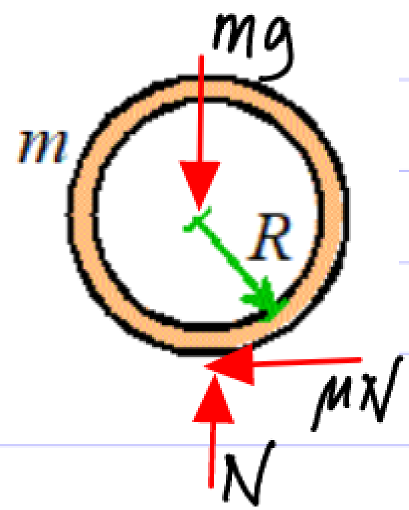
(4) The velocity and angular velocity when slip ceases



Slip $\Rightarrow$ we need to find $\underline{v}_c$ to draw correct FBD

Kinematics: $\underline{v}_c - \underline{v}_G = \omega_0 \underline{k} \times (\underline{r}_c - \underline{r}_G)$

$\Rightarrow \underline{v}_c = (V + \omega_0 R)\,\underline{i} \quad \Rightarrow v_{cx} > 0$



$\underline{F} = m\underline{a}_G \Rightarrow -\mu N \underline{i} + (N - mg)\underline{j} = m a_{Gx} \underline{i}$

Angular momentum about G

$$-\mu N R = \cancelto{=0}{\underline{r}_G} \times m \underline{a}_G + I_{Gzz} \alpha_z \underline{k}$$

Hence $N = mg$

$$a_{Gx} = -\mu g$$

$$\alpha_z = -\frac{\mu m g R}{I_{Gzz}} \qquad I_{Gzz} = mR^2 \text{ for ring}$$

$$\Rightarrow \quad \alpha_z = -\mu g / R$$

(1), (2) Use constant accel formulas

$$\boxed{v_{Gx} = V - \mu g t \qquad \omega = \omega_0 - \frac{\mu g}{R} t}$$

(3) Slip continues until $\underline{v_c} = \underline{0}$

$$v_{cx} = v_{Gx} + \omega R \Rightarrow V + \omega_0 R - 2\mu g t$$

$\Rightarrow$ at end of slip $v_{cx} = 0 \Rightarrow$ $\boxed{t = \dfrac{V + \omega_0 R}{2\mu g}}$

(4) $$v_{Gx} = V - \mu g t = V - \frac{(V + \omega_0 R)}{2}$$

$$\Rightarrow \boxed{v_{Gx} = \frac{(V - \omega_0 R)}{2}}$$

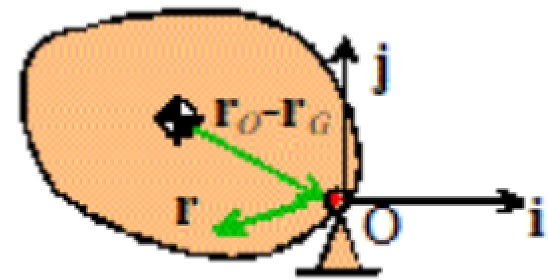
$$\omega = \omega_0 - \frac{\mu g}{R} t = \omega_0 - \frac{V + \omega_0 R}{2R}$$

$$\Rightarrow \boxed{\omega = \frac{1}{2}\left(\omega_0 - \frac{V}{R}\right)}$$

Note direction reverses if $\omega_0 > \frac{V}{R}$

6.5.8 Simplified method for fixed axis rotation

For an object rotating about a fixed point we can use 2 methods:



(A) General Formulas (Sect 6.5.2)

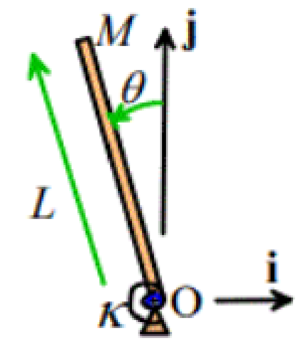
(B) (i) Find I_o with parallel axis theorem
(ii) Use special formula for angular momentum $\underline{h}_o = I_o \underline{\omega}$

$$\boxed{\sum \underline{r}\times\underline{F} + \sum \underline{Q} = I_o \underline{\alpha} + \underline{\omega}\times(I_o \underline{\omega}) \qquad 3D}$$

$$\boxed{\sum \underline{r}\times\underline{F} + \sum Q_z \underline{k} = I_{ozz}\, \alpha_z \underline{k} \qquad 2D}$$

NB: take moment about O

Example 6.5.9: A bar with mass *M* and length *L* is stabilized in its vertical configuration by a torsional spring with stiffness κ
Find a formula for its natural frequency of vibration.

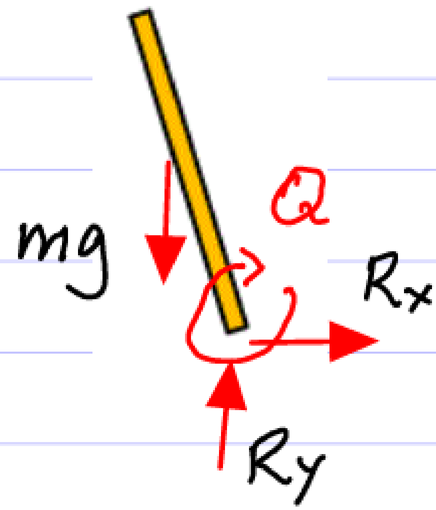


Approach: (1) EOM for θ (fixed axis rotation)
(2) Use vibration formulas

Parallel axis theorem $\quad I_0 = mL^2/12 + m(L/2)^2 = mL^2/3$

Angular momentum about O

$$-Q + mg\frac{L}{2}\sin\theta = I_{Ozz}\alpha_z = I_{Ozz}\frac{d^2\theta}{dt^2}$$



Spring formula $\quad Q = k\theta$

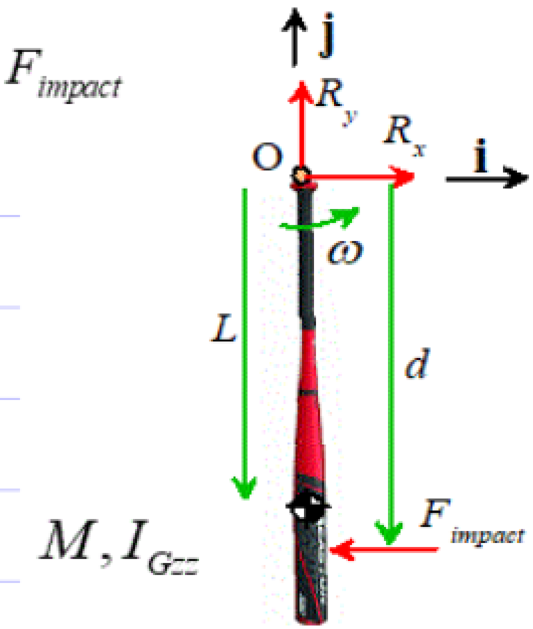
Small angle approx $\quad \sin\theta \approx \theta$

$$\Rightarrow \frac{mL^2}{3}\frac{d^2\theta}{dt^2} = -k\theta + \frac{mgL}{2}\theta$$

$$\Rightarrow \frac{1}{\omega_n^2} \quad \frac{2mL^2}{3(2k - mgL)}\frac{d^2\theta}{dt^2} + \theta = 0$$

$$\boxed{\omega_n = \sqrt{\frac{3(2k - mgL)}{2mL^2}}}$$

Example 6.5.10: The bat rotates about O. It is subjected to a force F_{impact}. Find a formula for the value of *d* that makes $R_x = 0$



$$\underline{F} = m\underline{a}_G \Rightarrow (R_x - F_{impact})\underline{i} + R_y\underline{j} = m\underline{a}_G$$

Angular momentum about O

$$-F_{impact}\, d = I_{Ozz}\, \alpha_z \underline{k}$$

$$\Rightarrow \alpha_z = -F_{impact}\, d / I_{Ozz}$$

Kinematics $\quad \underline{a}_G - \underline{a}_O = \alpha_z \underline{k} \times (-L\underline{j}) + \omega^2 L \underline{j}$

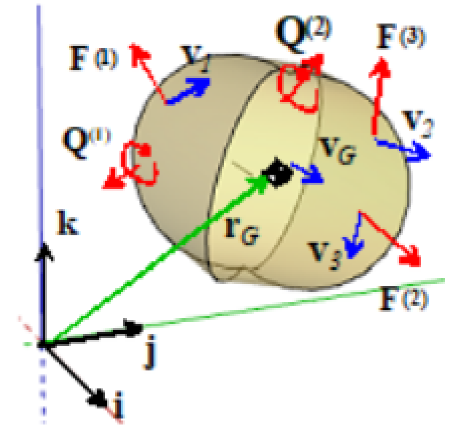
$$\Rightarrow \underline{a}_G = \alpha_z L \underline{i} + \omega^2 L \underline{j}$$

Hence $\quad R_x = F_{impact} + m\alpha_z L = F_{impact}\left(1 - \dfrac{d m L}{I_{Ozz}}\right)$

For $R_x = 0 \Rightarrow$ $\boxed{d = \dfrac{I_{Ozz}}{mL}}$

Deriving the angular momentum formula

$$\boxed{\text{Prove that } \sum \mathbf{r}\times\mathbf{F}+\sum\mathbf{Q}=\mathbf{r}_G\times M\mathbf{a}_G+\mathbf{I}_G\boldsymbol{\alpha}+\boldsymbol{\omega}\times\left(\mathbf{I}_G\boldsymbol{\omega}\right)}$$



(1) Moment-angular momentum relation

$$\sum \mathbf{r}\times\mathbf{F}+\sum\mathbf{Q}=\frac{d\mathbf{h}}{dt} \qquad \mathbf{h}=\mathbf{r}_G\times M\mathbf{v}_G+\mathbf{I}_G\boldsymbol{\omega}$$

(2) Recall $\displaystyle \frac{d\mathbf{I}_G}{dt}=\mathbf{W}\mathbf{I}_G-\mathbf{I}_G\mathbf{W}$

(3) Hence $$\frac{d\mathbf{h}}{dt}=\underbrace{\boxed{\frac{d\mathbf{r}_G}{dt}}}_{\mathbf{v}_G}\times M\mathbf{v}_G+\mathbf{r}_G\times M\underbrace{\boxed{\frac{d\mathbf{v}_G}{dt}}}_{\mathbf{a}_G}+\boxed{\frac{d\mathbf{I}_G}{dt}}\boldsymbol{\omega}+\mathbf{I}_G\underbrace{\boxed{\frac{d\boldsymbol{\omega}}{dt}}}_{\boldsymbol{\alpha}}$$

(4) Recall $\mathbf{W}\mathbf{u}=\boldsymbol{\omega}\times\mathbf{u} \quad \forall\mathbf{u}$

$$\frac{d\mathbf{I}_G}{dt}\boldsymbol{\omega}=\left(\mathbf{W}\mathbf{I}_G-\mathbf{I}_G\mathbf{W}\right)\boldsymbol{\omega}=\boldsymbol{\omega}\times\left(\mathbf{I}_G\boldsymbol{\omega}\right)-\mathbf{I}_G\left(\boldsymbol{\omega}\times\boldsymbol{\omega}\right)=\boldsymbol{\omega}\times\left(\mathbf{I}_G\boldsymbol{\omega}\right)$$

(5) So... $$\boxed{\sum \mathbf{r}\times\mathbf{F}+\sum\mathbf{Q}=\mathbf{r}_G\times M\mathbf{a}_G+\mathbf{I}_G\boldsymbol{\alpha}+\boldsymbol{\omega}\times\left(\mathbf{I}_G\boldsymbol{\omega}\right)}$$